

# Precalculus

Precalculus combines the trigonometric, geometric, and algebraic techniques needed to prepare students for the study of calculus, and strengthens students' conceptual understanding of problems and mathematical reasoning in solving problems. Facility with these topics is especially important for students intending to study calculus, physics, and other sciences, and/or engineering in college. Because the standards for this course are (+) standards, students selecting this Precalculus course should have met the college and career ready standards.

For the Precalculus course, instructional time should focus on four critical areas: (1) extend work with complex numbers; (2) expand understanding of logarithms and exponential functions; (3) use characteristics of polynomial and rational functions to sketch graphs of those functions; and (4) perform operations with vectors.

- 1) Students continue their work with complex numbers. They perform arithmetic operations with complex numbers and represent them and the operations on the complex plane. Students investigate and identify the characteristics of the graphs of polar equations, using graphing tools. This includes classification of polar equations, the effects of changes in the parameters in polar equations, conversion of complex numbers from rectangular form to polar form and vice versa, and the intersection of the graphs of polar equations.
- 2) Students expand their understanding of functions to include logarithmic and trigonometric functions. They investigate and identify the characteristics of exponential and logarithmic functions in order to graph these functions and solve equations and practical problems. This includes the role of  $e$ , natural and common logarithms, laws of exponents and logarithms, and the solutions of logarithmic and exponential equations. Students model periodic phenomena with trigonometric functions and prove trigonometric identities. Other trigonometric topics include reviewing unit circle trigonometry, proving trigonometric identities, solving trigonometric equations, and graphing trigonometric functions.
- 3) Students investigate and identify the characteristics of polynomial and rational functions and use these to sketch the graphs of the functions. They determine zeros, upper and lower bounds,  $y$  - intercepts, symmetry, asymptotes, intervals for which the function is increasing or decreasing, and maximum or minimum points. Students translate between the geometric description and equation of conic sections. They deepen their understanding of the Fundamental Theorem of Algebra.
- 4) Students perform operations with vectors in the coordinate plane and solve practical problems using vectors. This includes the following topics: operations of addition, subtraction, scalar multiplication, and inner (dot) product; norm of a vector; unit vector; graphing; properties; simple proofs; complex numbers (as vectors); and perpendicular components.

The Standards for Mathematical Practice complement the content standards so that students increasingly engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle, and high school years.

Note: The source of this introduction is the *Massachusetts Curriculum Framework for Mathematics* (Malden: Massachusetts Department of Elementary and Secondary Education, 2011), 155.

# Precalculus Overview

## Number and Quantity

### The Complex Number System

- Perform arithmetic operations with complex numbers.
- Represent complex numbers and their operations on the complex plane.

### Vector and Matrix Quantities

- Represent and model with vector quantities.
- Perform operations on vectors.
- Perform operations on matrices and use matrices in applications.

## Algebra

### Seeing Structure in Expressions

- Interpret the structure of expressions.

### Arithmetic with Polynomials and Rational Expressions

- Rewrite rational expressions.

### Creating Equations

- Create equations that describe numbers or relationships.

### Reasoning with Equations and Inequalities

- Solve systems of equations.

## Functions

### Interpreting Functions

- Interpret functions that arise in applications in terms of the context.
- Analyze functions using different representations.

### Building Functions

- Build new functions from existing functions.

### Trigonometric Functions

- Expand the domain of trigonometric functions using a unit circle.
- Model periodic phenomena with trigonometric functions.
- Prove and apply trigonometric identities.

### Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

## Geometry

Similarity, Right Triangles, and Trigonometry

- Apply trigonometry to general triangles.

Expressing Geometric Properties with Equations

- Translate between the geometric description and the equation for a conic section.

## Number and Quantity

### The Complex Number System

N – CN

Perform arithmetic operations with complex numbers.

- (+) Find the conjugate of a complex number; use conjugates to find moduli and quotients of complex numbers.

Represent complex numbers and their operations on the complex plane.

- (+) Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.
- (+) Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation. *For example,  $(-1 + \sqrt{3}i)^3 = 8$  because  $(-1 + \sqrt{3}i)$  has modulus 2 and argument  $120^\circ$ .*
- (+) Calculate the distance between numbers in the complex plane as the modulus of the difference, and the midpoint of a segment as the average of the numbers at its endpoints.

### Vector and Matrix Quantities

N – VM

Represent and model with vector quantities.

- (+) Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g.,  $v$ ,  $|v|$ ,  $\|v\|$ ,  $v$ ).
- (+) Find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point.
- (+) Solve problems involving velocity and other quantities that can be represented by vectors.

Perform operations on vectors.

- (+) Add and subtract vectors.
  - Add vectors end-to-end, component-wise, and by the parallelogram rule. Understand that the magnitude of a sum of two vectors is typically not the sum of the magnitudes.
  - Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum.
  - Understand vector subtraction  $v - w$  as  $v + (-w)$ , where  $-w$  is the additive inverse of  $w$ , with the same magnitude as  $w$  and pointing in the opposite direction. Represent vector subtraction graphically by connecting the tips in the appropriate order, and perform vector subtraction component-wise.
- (+) Multiply a vector by a scalar.
  - Represent scalar multiplication graphically by scaling vectors and possibly reversing their direction; perform scalar multiplication component-wise, e.g., as  $c(v_x, v_y) = (cv_x, cv_y)$ .

- b. Compute the magnitude of a scalar multiple  $c\mathbf{v}$  using  $\|c\mathbf{v}\| = |c|\mathbf{v}$ . Compute the direction of  $c\mathbf{v}$  knowing that when  $|c|\mathbf{v} \neq 0$ , the direction of  $c\mathbf{v}$  is either along  $\mathbf{v}$  (for  $c > 0$ ) or against  $\mathbf{v}$  (for  $c < 0$ ).

Perform operations on matrices and use matrices in applications.

6. (+) Use matrices to represent and manipulate data, e.g., to represent payoffs or incidence relationships in a network.
7. (+) Multiply matrices by scalars to produce new matrices, e.g., as when all of the payoffs in a game are doubled.
8. (+) Add, subtract, and multiply matrices of appropriate dimensions.
9. (+) Understand that, unlike multiplication of numbers, matrix multiplication for square matrices is not a commutative operation, but still satisfies the associative and distributive properties.
10. (+) Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse.
11. (+) Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors.
12. (+) Work with  $2 \times 2$  matrices as transformations of the plane, and interpret the absolute value of the determinant in terms of area.

## Algebra

### Seeing Structure in Expressions

A – SSE

Interpret the structure of expressions.

1. Interpret expressions that represent a quantity in terms of its context. ★
  - a. Interpret parts of an expression, such as terms, factors, and coefficients.
  - b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret  $P(1 + r)n$  as the product of  $P$  and a factor not depending on  $P$ .*
2. Use the structure of an expression to identify ways to rewrite it. *For example, see  $x^4 - y^4$  as  $(x^2)^2 - (y^2)^2$ , thus recognizing it as a difference of squares that can be factored as  $(x^2 - y^2)(x^2 + y^2)$ .*

### Arithmetic with Polynomials and Rational Expressions

A – APR

Rewrite rational expressions.

6. Rewrite simple rational expressions in different forms; write  $a(x)/b(x)$  in the form  $q(x) + r(x)/b(x)$ , where  $a(x)$ ,  $b(x)$ ,  $q(x)$ , and  $r(x)$  are polynomials with the degree of  $r(x)$  less than the degree of  $b(x)$ , using inspection, long division, or, for the more complicated examples, a computer algebra system.

## PC Precalculus

- (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

### Creating Equations

A – CED

Create equations that describe numbers or relationships.

- Create equations and inequalities in one variable including ones with absolute value and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.* CA ★
- Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. ★
- Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.* ★
- Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law  $V = IR$  to highlight resistance  $R$ .* ★

### Reasoning with Equations and Inequalities

A – REI

Solve systems of equations.

- (+) Represent a system of linear equations as a single matrix equation in a vector variable.
- (+) Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension  $3 \times 3$  or greater).

## Functions

### Interpreting Functions

F – IF

Interpret functions that arise in applications in terms of the context.

- For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.* ★
- Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function  $h$  gives the number of person-hours it takes to assemble  $n$  engines in a factory, then the positive integers would be an appropriate domain for the function.* ★

**Analyze functions using different representations.**

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. ★
  - d. (+) Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior. ★
  - e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. ★
10. (+) Demonstrate an understanding of functions and equations defined parametrically and graph them. CA★
11. (+) Graph polar coordinates and curves. Convert between polar and rectangular coordinate systems. CA

**Building Functions**

F – BF

**Build new functions from existing functions.**

3. Identify the effect on the graph of replacing  $f(x)$  by  $f(x) + k$ ,  $k f(x)$ ,  $f(kx)$ , and  $f(x + k)$  for specific values of  $k$  (both positive and negative); find the value of  $k$  given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*
4. Find inverse functions.
  - b. (+) Verify by composition that one function is the inverse of another.
  - c. (+) Read values of an inverse function from a graph or a table, given that the function has an inverse.
  - d. (+) Produce an invertible function from a non-invertible function by restricting the domain.

**Trigonometric Functions**

F – TF

**Expand the domain of trigonometric functions using a unit circle.**

4. (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

**Model periodic phenomena with trigonometric functions.**

6. (+) Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.
7. (+) Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context. ★

**Prove and apply trigonometric identities.**

9. (+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.
10. (+) Prove the half angle and double angle identities for sine and cosine and use them to solve problems. CA★

**Geometry****Similarity, Right Triangles, and Trigonometry**

G – SRT

Apply trigonometry to general triangles.

9. (+) Derive the formula  $A = 1/2 ab \sin(C)$  for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.
10. (+) Prove the Laws of Sines and Cosines and use them to solve problems.
11. (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

**Expressing Geometric Properties with Equations**

G– GPE

Translate between the geometric description and the equation for a conic section.

3. (+) Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.
- 3.1 Given a quadratic equation of the form  $ax^2 + by^2 + cx + dy + e = 0$ , use the method for completing the square to put the equation in standard form; identify whether the graph of the equation is a circle, parabola, ellipse, or hyperbola, and graph the equation. CA